A non-spider counterexample to Li's coefficient-sign prediction

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Ethan Li asked whether a particular coefficient of the chromatic symmetric function must be negative for every odd-order tree of maximum degree four admitting connected partitions of every type. Li also singled out non-spiders and suggested that a partition into one triple and otherwise pairs should be unique. The tree below answers both questions in that narrower class. Li, §4, p. 16, discussion preceding Problem 4.6 and Problem 4.9.

Theorem. There is a 31-vertex non-spider tree T of maximum degree four with all 6,842 connected partition types. Its coefficient of $e_3e_2^{14}$ is $+29$, and it has **eight** connected partitions of type $(3, 2^{14})$.

1. The tree



Endpoint labels identify vertices. A segment represents a path of the indicated length.

For exact labels take the five paths

$$(0, 1), \quad (0, 2, 3), \quad (0, 4, 5, \ldots, 14), \quad (8, 15), \quad (0, 16, 17, \ldots, 30).$$

Their consecutive pairs are the edges. Vertex 0 has degree four and vertex 8 degree three. A spider has at most one vertex of degree greater than two. Removing all leaves still leaves vertex 0 with three neighbors, so this example also lies outside the caterpillar class.

A connected partition of type λ splits the vertices into connected blocks whose sizes are the parts of λ. In a tree, deleting the edges between blocks gives exactly that partition. For the eight partitions of type $(3, 2^{14})$, the triple is either $\{0, 1, 16\}$ or $\{j, j+1, j+2\}$ for $j = 16, 18, 20, 22, 24, 26, 28$. Removing any of these triples leaves a forest with a perfect matching, necessarily unique. Section 2 checks all triples and all types.

2. Verifying every partition type

Root the tree at 0. In a partly processed subtree, record a pair (a, λ), where a is the size of the component containing its root and λ lists the sizes of all the other components. A single vertex starts at $(1, \varnothing)$ with multiplicity one. When a child in state (b, μ) is attached to a parent in state (a, λ), the two choices give

Connecting edge	New state
Retained	$(a + b, \lambda \uplus \mu)$
Deleted	$(a, \lambda \uplus \mu \uplus \{b\})$

Multiply the parent and child counts and add the product to each displayed state. Every edge cut in the combined tree restricts uniquely to the two smaller cuts and the choice on the connecting edge. Induction therefore gives the exact number of cuts in each state. At the root, add a to the list of parts.

Total edge cuts counted	$2^{30} = 1073741824$
Distinct component-size partitions	6,842
Integer partitions of 31	6,842
Cuts producing one triple and fourteen pairs	8

The accompanying standard-library Python program independently generates all integer partitions of 31 in increasing part order. For **every** type, the recurrence retains a representative edge-cut mask. The program rebuilds all 6,842 representatives using disjoint sets and compares their actual component sizes with the requested parts. A second check lists the triples whose deletion leaves a perfect matching by stripping forced leaf edges.

3. The sign calculation

The chromatic symmetric function X_T sums one monomial per proper coloring, using one variable per color. The elementary symmetric functions e_j give a basis in which Li asks for the coefficient of $e_3 e_2^{14}$. Work modulo t^2 and give three colors weights $1 + \frac{t}{2}$, $-1 + \frac{t}{2}$, and $-t$, setting all other weights to zero. Direct multiplication gives

$$e_1 = 0, \quad e_2 = -1, \quad e_3 = t, \quad e_j = 0 \quad (j > 3).$$

By homogeneity, the linear term in X_T is precisely $[e_3 e_2^{14}] X_T$, since $(-1)^{14} = 1$. A coloring that uses the third color twice contributes zero. It therefore suffices to count ordinary two-colorings and colorings with one vertex assigned the third color.

4. Evaluating the linear term

Let A be the 16 vertices at odd distance from 0 and B the 15 at even distance. With no third color there are just two proper colorings. Their weights are

$$\left(1+\frac{t}{2}\right)^{16}\left(-1+\frac{t}{2}\right)^{15}+\left(1+\frac{t}{2}\right)^{15}\left(-1+\frac{t}{2}\right)^{16}=-t \bmod t^2.$$

If vertex v alone uses the third color, each component of $T-v$ has two choices of coloring by the first two colors. Its contributions at $t=0$ cancel if its order is odd. When every component has even order, swapping the two colors in any component preserves its sign. There are $2^{\deg(v)}$ choices. In the coloring with A positive and B negative, the remaining sign is -1 if $v\in A$ and $+1$ if $v\in B$. Multiplication by $-t$ gives the following contributions to the coefficient of t.

Class	Vertices whose deletion leaves only even components	Total
A, degree 1	1, 14, 30	$+6$
A, degree 2	10, 12, 16, 18, 20, 22, 24, 26, 28	$+36$
B, degree 1	3, 15	-4
B, degree 2	5, 7	-8

Adding the two-color contribution gives $-1+6+36-4-8=29$. □

Relation to the larger conjecture. The Dahlberg–She–van Willigenburg conjecture predicts that some elementary coefficient is negative for every tree of maximum degree at least four. In this example the coefficient of $e_2^8e_3^5$ is -6304. The verifier checks this by the power-sum expansion over edge cuts and, independently, by exact proper-coloring evaluations on five three-color alphabets. The example therefore excludes Li's proposed choice of coefficient as a universal way to prove the larger conjecture. Dahlberg, She and van Willigenburg.

Prior work. The unrestricted wording of Problem 4.9 already has a spider counterexample implicit in two sources. Horňák and Woźniak list the partitionable spider with arm orders $(2,3,13,16)$ including the common center. Its edge lengths are $(1,2,12,15)$. Zheng's two-odd-leg formula gives $4(0+7-1-6)+7=7$. The non-spider example above addresses the additional restriction Li singled out. The October 2026 theorem of Tang, Wang and Wang concerns caterpillars. Horňák–Woźniak, Table 2 · Zheng, Theorem 5.3 · Tang–Wang–Wang.

Priority check, 17 November 2026. Searches of the citing literature, arXiv, DOI records and public proof archives used Li's problem number, coefficient, non-spider restriction, and the older term "arbitrarily vertex decomposable." I found no earlier answer to the non-spider sign question. The attached `verify.py` reproduces the exact checks with `python3 verify.py`.