The limit-shape conjecture for generalized Bulgarian solitaire

Research note · 17 November 2026

Eriksson, Jonsson and Sjöstrand asked whether stable and recurrent configurations always approach the same shape. Their published Conjecture 1 allows arbitrary positive scaling factors. A two-cycle disproves it at constant scale. For every sequence of picking rules and every scale tending to infinity, we prove the conjectured implication. On each finite deck, shifting either diagram one column to the left puts it below the other. Original paper, §7.

A two-cycle that stays visibly different

Choose a set L of positive ranks containing 1. Each move takes the cards at those ranks from the bottom of every pile and collects them into a single new pile. A configuration is **stable** when one move fixes it, and **recurrent** when some positive number of moves returns it to itself. Write its pile sizes in decreasing order, with zeros appended.

For each deck size $n \geq 3$, choose $B = \lfloor \frac{n}{2} \rfloor + 1$ and pick the bottom B cards of every pile, taking a shorter pile in full. Then

$$(n) \to (B, n-B) \to (n).$$

The first move separates off B cards. Both piles are now small enough to be collected in full. For example, with 101 cards the two-cycle is $(101) \leftrightarrow (51, 50)$, while $(76, 25)$ is stable under exactly the same rule.

When no fixed point has exactly n cards, the authors take as reference the stable configuration ζ of largest mass at most n. In this example set $h = \lfloor \frac{n+B}{2} \rfloor$. The stable profile with largest pile h is $(h, h-B)$, of mass $s = 2h - B \in \{n-1, n\}$. Indeed, the successive piles of any fixed point with top j are $j, (j-B)_+, (j-2B)_+, \ldots$, whose total is strictly increasing in j. Here $B \leq h < 2B$ and the next top gives mass exceeding n.

Under the paper's scaling, a partition λ has profile

$$D_a(\lambda, x) = \frac{a}{|\lambda|} \lambda_{\lfloor ax \rfloor + 1}, \quad x > 0.$$

At the allowed scale $a_n = 1$, the stable profile tends to $\frac{3}{4}$ on $(0, 1)$, $\frac{1}{4}$ on $[1, 2)$, and zero on $[2, \infty)$. The recurrent configuration (n) has profile 1 on $(0, 1)$ and zero thereafter. Already at $x = \frac{1}{2}$ the limits differ. This disproves the statement in Definitions 2–4 and Conjecture 1.



Stable limit · heights ¾ and ¼



One-pile recurrent limit · height 1

The return time of a card

Fix a rule L and a deck of $n \geq 1$ cards. Let $\sigma(j)$ be the number picked from a pile of size j, and put $f(j) = j - \sigma(j)$, with $f(0) = 0$. Both functions are nondecreasing with successive increments zero or one. Every positive pile strictly shrinks between its creation and disappearance.

Lemma 1. Follow a cycle in both time directions. Write b_t for the size of the new pile at time t. For each birth rank r, the indicator of $b_t \geq r$ is periodic with period $\tau(r)$, where

$$\tau(r) = \begin{cases} 1 & \text{if } r \in L \\ 1 + \tau(f(r)) & \text{if } r \notin L \end{cases}.$$

Proof. A card born at rank r returns to the collected pile exactly $\tau(r)$ moves later. If skipped on its first move, it drops to rank $f(r)$, independent of the height above it. Let $M = \max_t b_t$. The ranks arriving at time t, recorded by their ranks at their previous births, form the set

$$A_t = \left\{ r \in \{1, \ldots, M\} : b_{t-\tau(r)} \geq r \right\}.$$

For each r only the birth at time $t - \tau(r)$ can supply that rank. Hence $|A_t| = b_t$. Among all sets of b_t positive ranks, the first b_t integers uniquely minimize their sum. Sum those rank sums over a full period. For each r, shifting time by $\tau(r)$ preserves its occurrence count, so the total for the arriving sets equals the total for the initial segments. Each individual inequality must be an equality. Thus A_t contains exactly the first b_t integers at every time. This gives the asserted periodicity. $\square$

Lemma 2. If $m = \min_t b_t$, then $f(M) \leq m$.

Proof. The claim is immediate when $m = M$. Every rank r in $(m, M]$ otherwise has a nonconstant presence indicator. Such a rank cannot be picked on the first move, since Lemma 1 would give period one. Hence σ is constant, with value $c = \sigma(m) \geq 1$, throughout $[m, M]$.

If $M - m > c$, take $r = m + 1$ and $s = r + c \leq M$. Since $f(s) = r$, their return times are $\tau(r)$ and $\tau(r) + 1$. The sets of times when the ranks are present are nested, as presence of s requires presence of r. Yet some time has s present and some time has r absent. By the Chinese remainder theorem a single time can realize these two residue classes modulo the two coprime return times. This is impossible. Thus $M - m \leq c$, and $f(M) = M - c \leq m$. $\square$

Interlacing. Write $S(j) = (j, f(j), f^2(j), \ldots)$, where superscripts denote iteration. These are precisely the stable configurations, by the fixed-point characterization in Lemma 1. On a cycle the age-k pile has size $f^k(b_{t-k})$, between $f^k(m)$ and $f^k(M)$. Lemma 2 gives $f^{k+1}(M) \leq f^k(m)$, so the age order already lists the piles in decreasing size. In particular $S(m) \leq \rho \leq S(M)$ coordinatewise for any recurrent ρ on this cycle.

The finite bound and its limit

Let $\zeta = S(j)$ be the authors' largest stable configuration by mass at most n. Since $|S(m)| \le n \le |S(M)|$ and stable mass is strictly increasing with its top, $m \le j \le M$. Both ζ and ρ therefore lie between the same two profiles. The one-column bound from Lemma 2 gives, for every $i \ge 1$,

$$\zeta_{i+1} \le \rho_i, \quad \rho_{i+1} \le \zeta_i.$$

This compares any recurrent state against the same stable reference, even when their masses differ.

Theorem. Let L_n be arbitrary admissible picking rules and $a_n \to \infty$. For every $\varepsilon > 0$, the distance between the normalized profiles of a recurrent state and the stable reference tends uniformly to zero on $x \ge \varepsilon$, uniformly over all recurrent states on n cards. Consequently any pointwise stable limit on $x > 0$ is the common recurrent limit there.

Proof. Write $s = |\zeta|$ and $d_i = \zeta_i - \zeta_{i+1} = \sigma(\zeta_i)$. These drops are nonincreasing. The first i piles contain at least $(1 + 2 + \ldots + i)d_i$ cards, whence $d_i \le 2\frac{s}{i(i+1)}$. Interlacing implies for $i \ge 2$

$$|\rho_i - \zeta_i| \le d_{i-1} \le \frac{2s}{i(i-1)}.$$

The mass estimate $0 \le n - s \le \sqrt{2n}$ follows from Lemma 3. For completeness, let ℓ be the length of $S(j)$. Increasing j by one adds at most one card at each of its ℓ positions, because f is 1-Lipschitz, and can add at most a single final 1. Thus $n - s < |S(j+1)| - s \le \ell + 1$. The strictly decreasing positive piles of $S(j)$ give $\frac{\ell(\ell+1)}{2} \le s \le n$.

Now take $a = a_n$, $x \ge \varepsilon$ and $a\varepsilon \ge 2$. With $i = \lfloor ax \rfloor + 1$, the change of normalization from s to n contributes at most $\left(\frac{a}{i}\right)\left(1 - \frac{s}{n}\right)$, since $\zeta_i \le \frac{s}{i}$. Together these estimates give

$$|D_a(\rho, x) - D_a(\zeta, x)| \le \frac{4}{a\varepsilon^2} + \frac{\sqrt{2}}{\varepsilon\sqrt{n}}.$$

Both terms vanish uniformly in the rule, the recurrent state and $x \ge \varepsilon$. $\square$

Consequence. In particular, this transfers the authors' entire family of interpolating stable shapes for proportional picking ($nq_n^2 \to C \in (0, \infty)$) to all recurrent configurations at the same scaling. Their Theorem 5(c) supplies the stable shape and a diverging scale in this regime. Theorem 5 and §9.

Source conventions and verification

The 2017 formulation. The authors' March 2017 preprint normalized each partition by its own largest pile, giving both unit area and unit height. In §7.1 they suggest comparing the return periods of cards born at different ranks. Lemmas 1 and 2 give an exact form of this idea. The example on page 1 also disproves the preprint's version. For the stable two-pile profile on page 1, the normalized heights are 1 and $\frac{h-B}{h} \to \frac{1}{3}$. Its column widths are $\frac{h}{s} \to \frac{3}{4}$. Since $B > \frac{n}{2}$ gives $h < 3\frac{B}{2}$, the widths approach $\frac{3}{4}$ strictly from above, so the stable functions converge even at their two limiting jumps. The recurrent profile from (n) has width and height 1. At $x = \frac{7}{8}$ its value is 1, while the stable value tends to $\frac{1}{3}$.

Earlier results. The 2020 journal paper uses a common horizontal scaling sequence. Definition 3 requires positive factors. Its Theorem 4 separately assumes that they diverge. Conjecture 1 is stated under the common-scale definitions. For proportional picking, §9 proves agreement in the triangular and exponential regimes. Our fine-scale theorem applies to all choices of layers. Brian Hopkins' 2024 Theorem 5 had already classified cycles when the rule picks the bottom B cards. The two-cycle used here also follows from his classification. The limit-shape counterexample and the interlacing argument for arbitrary layer sets are the claims of this note.

Check against the finite game. The Python 3 verifier attached to this PDF builds the actual map on integer partitions by taking the selected cards. It independently finds the directed cycles and compares their pile sizes with the largest admissible fixed point. It tests the rank-period identities of Lemma 1, the range bound of Lemma 2, and both interlacing inequalities.

Exhaustive check for decks of 1 to 14 cards	Count
Admissible rules, counted per deck size	16,383
Pairs of a rule and a partition	1,768,137
Directed cycles	20,531
Recurrent states counted across rules	48,426

It separately checks the counterexample formulas at every deck size from 1 through 1,000. Run `python3 verify.py` to reproduce these counts. The proofs on pages 1–3 apply at all deck sizes without computation.

Status. This is a local, unrefereed proof. Searches through 17 November 2026 for the authors' conjecture, later citing work, and cycle classifications for general L-solitaire found no earlier resolution of the full limit-shape question. The author attribution refers specifically to Conjecture 1 of the linked paper.